Computational Search of Long Skew-symmetric Binary Sequences with High Merit Factors

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Abstract

In this paper, we present a computational search for best-known merit factors of longer binary sequences with an odd length. Finding low autocorrelation binary sequences with optimal or suboptimal merit factors is a very difficult optimization problem. An improved version of the heuristic algorithm is presented and tackled to search for aperiodic binary sequences with good autocorrelation properties. A high-performance computation equipment was used in experiments to search skew-symmetric binary sequences with high merit factor values. After experimental work, as results, we present new binary sequences with odd lengths between 201 and 303 that are skew-symmetric and have the merit factor F greater than 8.5. Moreover, an example of a binary sequence having $F > 8$ has been found for all odd lengths between 201 and 303. The longest binary sequence with $F > 9$ found to date is of length 255.

1 Introduction

Binary sequences with low autocorrelation function properties are important in many areas, such as communication engineering [44, 46, 43, 45] and in statistical mechanics [34, 3, 32, 42]. Also in mathematics, this problem (see *Littlewood polynomial*) has attracted sustained interest [23, 26, 22]. Long binary sequences are essential for various applications of the coded exposure process [29, 30].

Finding Low Autocorrelation Binary Sequences (LABS) with optimal/good merit factors or peak sidelobe level is a challenging optimization problem. There are two types of binary sequences: periodic and aperiodic. In this paper, we are dealing with aperiodic ones.

A binary sequence $S = (s_0, s_1, \ldots, s_{L-1})$ has all entries either $+1$ or -1. Here, L denotes the sequence length. The aperiodic autocorrelation of S at shift k is defined as:

$$C_k(S) = \sum_{i=0}^{L-k-1} s_i s_{i+k},$$
$$\text{for } k = -(L-1), \ldots, -1, 0, 1, \ldots, L-1, \quad (1)$$

and the Integrated Sidelobe Level (ISL) metric of S is:

$$\text{ISL}(S) = \sum_{k=1}^{L-1} |C_k(S)|^2. \quad (2)$$

Note that $\text{ISL}(S)$ is defined as the sum of the squares of all off-peak autocorrelations (i.e., $k \neq 0$).

The LABS problem involves assigning values to the s_i that minimize $\text{ISL}(S)$ or maximize the *merit factor* $F(S)$ [20]:

$$F(S) = \frac{L^2}{2 \cdot \text{ISL}(S)}. \quad (3)$$

The merit factor $F(S)$, shortly F in the remaining of the paper, is a measure of the quality of the sequence in terms of engineering applications [5].

The *skew-symmetric sequences* have odd length with $L = 2n - 1$ and satisfy:

$$s_{n+i} = (-1)^i s_{n-i}, \; i = 1, 2, \ldots, n - 1. \quad (4)$$

which implies that $C_k(S) = 0$ for all odd k. The restriction of the problem to skew-symmetric sequences reduces the sequence's effective length from L up to approximately $L/2$. It means that the dimension of the problem and the search space are reduced. The search space is reduced from 2^L to approximately $2^{(L/2)}$ [34]. Note that the optimal skew-symmetric solutions might not be optimal for the whole search space.

Besides the merit factor, another metric for the LABS problem is the Peak Sidelobe Level, $\text{PSL}(S) = \max_{k=1}^{L-1} |C_k(S)|$ [28]. Most of the time, a sequence with the optimal PSL has a merit factor which is much lower than the optimal merit factor, and vice versa. In this paper, our key focus is to search for long aperiodic binary sequences with high merit factors. A reader interested in optimization of the PSL values is referred to works [15, 8, 12, 16, 9].

One of the main challenges when solving the LABS problem using the incomplete search is how to im-

plement a calculation of energy in Eq. (2) efficiently. Researchers developed efficient implementations of the energy calculation [19, 24, 6, 33, 14, 15]. They can be divided into two groups. The first group of the implementations, presented in [14, 6], utilizes an one-dimensional array to stores C_k values (see Eq. 1) and updates them when one bit flip is applied. The second group of the implementations, used in [19, 7] and also this work, store precalculated values of $s_i s_{i+k}$ (addends in Eq. (1)) in two two-dimensional arrays. Note that a similar efficient calculation using one-dimensional array can be also applied to finding a skew-symmetric solution of the odd length problem instances [13].

For the time being, aperiodic binary sequences with currently known best merit factors for lengths from 191 up to 225 are published in [7]. All these sequences are skew-symmetric with $8.6394 < F < 9.5851$. For lengths longer than 225 up to 301, there are known sequences for some lengths only and all of them have $F < 8$ (see collection [6]). Searching binary sequences, general or skew-symmetric, with a high merit factor, higher than 8 for a length longer than 230 is a challenging optimization problem.

Nowadays, parallel computation can be applied to tackle hard optimization problems. The power of multiple computers, which are not necessarily placed in the same location but can be distributed, is combined to solve multiple real-world problems. Grid computing is used in literature to make computations for finding (binary) sequences [36, 31, 6, 7].

In this paper, we use an improved version of the xLastovka [7] stochastic algorithm for searching skew-symmetric binary sequences. In particular, we investigate through extensive experimental runs the influence of dimensionality of binary sequences for odd lengths $225 < L \leq 303$. At the end of this experimental work, we are able to find a number of binary sequences that have merit factors higher than 8. The main contributions in this paper can be summarized as follows:

- The improved version of the algorithm has found skew-symmetric binary sequences with the same or better merit factor than previous algorithms for lengths between 201 and 225.

- For all lengths between 227 and 303, including, we have been able to find skew-symmetric binary sequences with $F > 8$.

- Examples are now known of binary sequences with $201 \leq L \leq 281$ and $L = 285$ having merit factors greater than 8.5.

- The longest skew-symmetric binary sequence with $F > 9$ found to date is of length $L = 255$.

The rest of our paper is organized as follows. The background is given in Section 2, where related work is also presented. In Section 3 an algorithm for solving a LABS problem in the sense of searching skew-symmetric binary sequences with high merit factor values is presented. Section 4 is the main part of the paper, where experimental results are conducted, and a brief discussion is given. Finally, the paper ends with a conclusion and future work in Section 5.

2 Background

Theoretical considerations from Golay in 1982 [21] give an upper bound on F of approximately 12.3248 as $L \to \infty$. However, Golay does not prove that 12.3248 is an upper bound on the asymptotic merit factor, because it relies on an unproven heuristic argument.

Owing to the practical importance and widespread applications of sequences with good autocorrelation properties, in particular with low peak sidelobe level values or large merit factor values, a lot of effort has been devoted to identifying these sequences either by analytical construction methods or computational approaches in the literature [40, 46, 41].

The *construction method* is set by so called appended rotated Legendre sequences with an asymptotic merit factor of 6.342061... [27, 26]. On the other hand, [1] used the modified Jacobi sequences together with the steep descent algorithm, and got an approximate asymptotic merit factor of 6.4382. The gap toward Golay's upper bound, i.e., 12.3248, still remains huge. Notice, that the study of the merit factor is fundamentally concerned with an asymptotic behavior, and not the identification of a particular sequence with a large merit factor. Nevertheless, J. Jedwab in the survey [25] gave a personal selection of challenges concerning the Merit Factor problem, arranged in order of increasing significance. The first challenge is as follows: "*Find a binary sequence S of length L > 13 for which F ≥ 10.*" Interestingly, in 2005, R. Ferguson and J. Knauer [18] suspected that in lengths of perhaps 250 for skew-symmetric sequences that merit factors $F > 10$ will regularly start to appear. To find a general or skew-symmetric binary sequence with $F \geq 10$ still remains open.

The search space of the LABS problem is of size 2^L. To locate good (optimal) solutions, two approaches exist: *complete* and *incomplete* search. The complete, or exact search, is able to find the optimal sequence, but it is unlikely to scale up to large sequences. The incomplete, or stochastic search, can obtain a result that may be optimal or close to optimal, i.e., it does not guarantee optimality.

Currently, the *optimal solutions* for binary sequences of even and odd lengths are known for $L \leq 66$, calculated by T. Packebusch and S. Mertens in 2016 [36]. Interestingly, it took 20 years to prove optimality for six sequences with $61 \leq L \leq 66$. Optimal solutions were proved by using the branch-and-bound algorithm.

Following the theoretical minimum energy level analysis, a new asymptotic merit factor value of 10.23 was *estimated* by Ukil [43] in 2015 based on sequences of length 4 to 60, found by the exhaustive search.

The *optimal skew-symmetric solutions* are known for $L \leq 119$ [36]. The previous record was $N \leq 89$ [37].

On the other hand, heuristic algorithms were introduced for solving many real-world problems. A heuristic algorithm can solve small instances easily and performs reasonably when tackling larger instances by quickly finding solutions that are close to optimal. The optimality of the solution is not guaranteed as in, for example, the exhaustive search.

Different techniques have been utilized to tackle the LABS problem, such as enumeration [10], evolution strategy [10], genetic algorithm [38], local search algorithm [17], branch and bound [34, 36], evolutionary algorithm with a suitable mutation operator [35], tabu search [24], directed stochastic algorithm [5], evolutionary algorithm [11], memetic algorithm combined with tabu search [19], and self-avoiding walk technique [6, 7].

The memetic agent-based paradigm [48], which combines evolutionary computation and local search techniques using parallel GPU implementation, is one of the promising meta-heuristics for solving a LABS problem.

Figure 1 shows the normalized aperiodic autocorrelation function (NAAF) in dB, i.e., $20\log_{10}\frac{C_k(S)}{L}$, of two binary sequences of length 213. One is randomly generated with $F = 1.3572$, and another has $F = 9.5393$, which is currently the best-known merit factor for sequences with a length over 200. The NPSL values, i.e., $20\log_{10}\frac{\text{PSL}(S)}{L}$, of the randomly generated sequence and the sequence with $F = 9.5393$ are -12.42 dB, and -25.74 dB, respectively, and the optimized sequence has the NPSL value, which is more than 13 dB lower than that of the starting sequence.

As we already said, a sequence with the optimal PSL usually has a merit factor that is much lower than the optimal merit factor, and vice versa. For example, the sequence with $F = 9.5393$ (Fig. 1) has PSL = 11, while a sequence with the "good" PSL value of 9 has the merit factor $F = 4.8386$, as reported in [8].

Notice, that the lssOrel [7] algorithm belongs to a group of the deep first search algorithms (after flipping each s_i it continues with the best sequence from s_i only), and it applies restarts (randomly initialized new starting sequence) after a predefined number of deep first search steps.

M. Dimitrov et al. [13], recently proposed an algorithm with a similar technique as used by lssOrel, with the difference that in [13], the algorithm starts a new search with the sequence of flipped s_i immediately, when s_i improves the merit factor. This algorithm applies small perturbations of flipping a few bits to continue the search after all flipped s_i did not improve the merit factor. It uses an efficient calculation of merit factors by the storing pre-calculated values of C_k in a one-dimensional array. This algorithm was able to perform a computational search on skew-symmetric sequences with lengths up to $10^5 + 1$ and obtained merit factors $F \approx 5$. For sequences with $200 < L < 300$, there are no reported sequences with merit factors greater than 7; the sequence with $F = 6.5319$ is reported for $L = 449$ [13]. Therefore, finding examples

of aperiodic binary sequences with $200 \leq L \leq 300$ that have merit factors greater than 8 or even greater than 9 is very challenging.
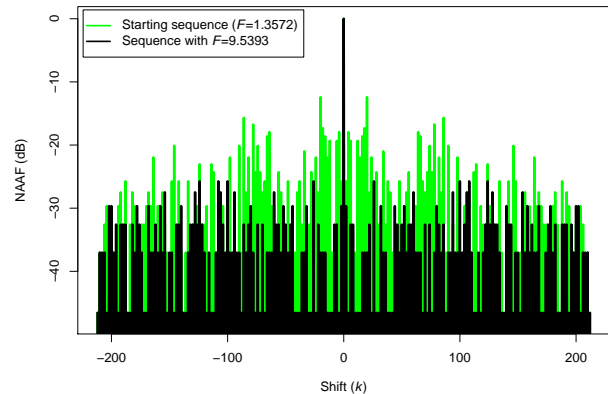

Figure 1: A binary sequence of length $L = 213$ with $F = 9.5393$ and a corresponding starting binary sequence.

3   Algorithm impxLast

In this section, we give an overview of the algorithm that was used in experimental work in this paper. We present the algorithm's features that are necessary for solving the LABS problem effectively and efficiently. In the experimental work, we used an improved version of the xLastovka [7] algorithm. We will call the improved version of the algorithm, impxLast. There are differences in the implementation of both algorithms. The xLastovka algorithm uses two hash tables, one for storing visited sequences, i.e., we say that a sequence is visited when its merit factor is calculated (Eq. (3) in general), and another for storing visited one-bit flipped neighborhood sequences. The main reason for using the second hash table lies in avoiding the already visited sequences and unnecessary calculating of their merit factors more times, since the LABS problem with skew-symmetric search space is very likely making cycles (i.e. repeating of already visiting sequences) when searching neighborhood sequences.

In the implementation of our improved algorithm, we omit the second hash table and, consequently, the impxLast algorithm requires more calculations of the merit factor values in comparison to the original algorithm. On the other hand, a new algorithm becomes slightly faster in the sense if we are focusing on the speed of both algorithms. The speed is defined as the number of function evaluations per second. A one-bit flip operation is counted as one function evaluation.

The mentioned difference between both algorithms plays a key role in making the new algorithm more successful in finding longer binary sequences with larger merit factor values. A greater amount of frequent accesses to the hash table can also reduce the performance of the algorithm. In our algorithm, we store

one-bit flipped neighborhood sequences into a priority queue based on their merit factors, i.e., the sequences with promising merit factors are inserted into a priority queue. Since the priority queue has a fixed size, a sequence with the worst merit factor is removed from the priority queue when a new sequence with higher merit is inserted into the priority queue. The same mechanism of storing one-bit flipped neighborhood sequences is also applied in the xLastovka algorithm.

A pseudocode of the impxLast algorithm is presented in Algorithm 1. It can be viewed as the best first search algorithm. In the main loop in our algorithm, each s_i is flipped, and its merit factor is calculated, then this new sequence is inserted into a priority queue, and, finally, it is compared to the sequence with the currently best merit factor. If necessary, the algorithm saves the new best sequence and its merit factor. After each s_i is flipped, the algorithm removes the sequence with the highest merit factor from the priority queue and continues the search process using this sequence as a new starting sequence. At the end of the search process, the algorithm outputs saved the best skew-symmetric binary sequence and its merit factor value.

A fast calculation merit factor of a sequence with one-bit flipped is used in Step 11 in Alg. 1. This mechanism was proposed in [19] and it uses two-dimensional structure for the efficient calculation of a merit factor when one bit is being changed. The same mechanism is also applied in [7]. This mechanism is very suitable for not very long binary sequences, i.e. L up to a few thousand, since its space complexity is $O(L^2)$. Note that Dimitrov et al. [13] also proposed an efficient mechanism for one-bit flip of a skew-symmetric binary sequence, which has space and time complexity $O(L)$ and it is suitable for searching very long binary sequences. We refer to [16, 8] for a couple of recent developments in searching of binary sequences with low PSL. In Step 6, $(L+1)/2$ flips are performed for each bit of a skew-symmetric binary sequence. The outer while loop that starts in Step 5 is executed while stopping criteria are not met. In the experimental work, we used a time of four days as the stopping condition in the impxLast algorithm.

4 Results and Discussion

In this section, we present our main results with discussion. The result for $201 \leq L \leq 303$ obtained by the impxLast algorithm are presented in Table 1, and the merit factors of the obtained binary sequences are plotted in Figure 2. In Table 1, the sequences obtained by the xLastovka [7] algorithm are marked with †, and the impxLast algorithm was able to find a sequence with the same merit factor value, too.

Figure 2 shows that all currently best-known merit factor values for skew-symmetric binary sequences with the odd length between 201 and 303 are greater than 8. With a construction method based on rotated Legendre sequences, one can construct a sequence with any arbitrary length (usually these lengths are required to

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Require: L ... length of sequence
Ensure: S_{best} ... best sequence found during optimization search
Ensure: F_{best} ... merit factor value of S_{best}
1: $S \leftarrow$ initialize a starting skew-symmetric sequence of length L
2: $F \leftarrow$ calculate merit factor using Eq. (3)
3: Insert S into hash table HT
4: $S_{best} \leftarrow S$; $F_{best} \leftarrow F$
5: **while** stopping criteria are not met **do**
      ** search neighborhood **
6:    **for** each $i \in (L+1)/2$ **do**
7:      $S_f \leftarrow$ flip s_i in S
8:      **if** $S_f \in$ HT **then**
9:        **continue** ** skip if sequence S_f was already visited **
10:      **end if**
11:      $F_f \leftarrow$ fast calculate merit factor of S_f (skew-symmetric)
12:      Insert S_f into a priority queue ordered by merit factor
13:      **if** $F_n > F_{best}$ **then**
      ** save the best sequence and its merit factor **
14:        $S_{best} \leftarrow S_f$; $F_{best} \leftarrow F_f$
15:      **end if**
16:    **end for**
      ** best first search **
17:    $S \leftarrow$ remove an item from the front of the queue
18: **end while**

Algorithm 1: Best first search algorithm (impxLast) with a priority queue.

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be prime numbers) that has a merit factor of approximately 6.34 [4, 1]. We can notice gab of 2 between currently best-known merit factors obtained by the impxLast algorithm and merit factors generated by the construction method on Figure 2.

The best-known merit factor values in Figure 2 are decreasing for longer sequences. A trend that the merit factor F decreases as length L increases can be interpreted in a way that a sequence with a high merit factor is harder to be found by a heuristic algorithm as its length increases. Note that our algorithm used the same amount of execution time for each sequence length. We used the SLING infrastructure [39] to perform one hundred runs for each LABS instance, and for some instances with lengths around 290, more executions were required to reach $F > 8$. Each run was limited to 4 days.

Figure 2 also illustrates how the best-known merit factor values have been changing over the last 35 years. The best-known merit factor was approximately 6 in 1985 [2, 3] for skew-symmetric sequences up to $L = 199$. Knauer's results are dated back to 2004, which is roughly speaking 2 decades after Beenker's results, and our results are approximately 2 decades after Knauer's results. This indicates how hard is the computational search of the LABS sequences with high merit factors.

Table 1: Merit factors and skew-symmetric binary sequences of length 201 up to 303 obtained by the impxLast algorithm. Sign † indicates the sequence and its merit factor that was also obtained in [7].

| L	F	Sequence	PSL
201	9.0993†	00FC7C04C5DF914630C9E3AF60A741258CB26FB95DECEAD6D4A	15
203	8.8927†	0E3C71E0783B8073005132B88CDDBF310553255BB5A56924B6D	13
205	8.7918	06333981E3DC3FDAA0FAF0E0106BAA4B414A01D52DD25A993326	15
207	9.1129†	0492402193C9DA4ED4207EED42740EEA5740EC789CB1975471C5	19
209	9.2544	071C7077C4519F3303F82A181F5F5A9A02952B3359BEED76B6DB6	15
211	9.0600†	0E38E0EF88A33E6607F054303EBEB534052A5666B37DDAED6DB6D	15
213	9.5393†	0545F75480D9EAD2791B136CB1E25B0C73B1B963C059CA8FF75EFE	11
215	8.9066	03E1B0FB2086FA3FE0628366088DDD6635F656AB7AE5D73AD396B5	13
217	8.9319	04A40097BF6B77C98C7BD94E876DC7684F9D16C98D77075178AAE0E	19
219	8.9580	03E0F983E0CDF0067745479A682475E79A4042265528CD6B59AD6B5	15
221	8.9584	00037B10994896495AC2928C7E79D9656C8382C1F8E788F98BB172AA	17
223	9.0383†	03E3E1F477C04FF98F31B71E403113546923932D9AAC54A24296B6B5	13
225	9.0144†	007030301EFF79C03DC6E7347AC71B6C16F3646DD2AD97545AB2B2B6A	15
227	8.8935	1CEE773970A8710C03C05C7AE04616456FA4854B54D125FD21B226EC9	15
229	8.8523	048B2681508906F64BA3F05CF4E0C8CA4F4DEB5210E746B88BFA86308E	13
231	8.7678	043FFF7E313D68635219C13C86706E5265CB149970365E0B136A2AAB45	15
233	8.9409	038E3387551EA3F15BC0D5911D4867668FDBB9FCAD1FB5205BFF6932492	15
235	8.9044	0DBB2327E024F9E6AE8AB42CF3C85405CB2CF43FDEFE69AC756A7373B8D	15
237	8.8039	03E24D9306D80F80E5C4082B0115D8C9DFBAB028AEDE4A94A9C6B39CE252	15
239	8.6678	1CD29A646F01F337DAEB55425A4204C5747874003EF8A332952E4679F0C9	13
241	8.6430	054AA555344A868922C49C36652531B1B3E3E672D8EC2388680EF3FFE00FE	15
243	8.6608	0383FE107E1E00F38E260034CCEFB8BDBAECCC355676DB2D5696A516AB5B5	17
245	8.7807	00502FC5B97D6B64C8863D348038918C9B892A8F3D2688CE707D791ED42BEA	13
247	8.6784	11810DC8CA02F4E19042E9658F71DE2F68922D861EF45196C2F57CDC8D1591	15
249	8.8573	000FF810253878C6360C380BD9EFDBB1B11D459D0A92CA726C9693E2BA954AA	17
251	8.7966	03B8E01DC783BE30F1B898A23FC4560356044AB77D9DB92D36BB5A48956DBB5	17
253	8.7206	0780041F0DE2791BFC10E4F2616C8C4828EC8C7A634E4BAD51B9625CB5AEAA96	13
255	9.0338	00FE00B692427B7F924094B3398FB9EE6E9BAD9B33C1D471AA3A7471E3D56AD5	19
257	8.9936	01B2CBE1F07DB2728B67E44A8C0730005EAAB36AC80EE567083631D6B5A50C31A	19
259	8.8056	1C3C6317188205030EBF1D99E819EE54BC06E995E99892BED350575D921364B49	17
261	8.7740	03F9F0E0184FB6387379B5C1522544E757F64EFE23FADF1973692714E9AA4B5952	15
263	8.5882	0C71C638FC545E3B2D6F22B4AA667F91FA91AA667FC3F72E0F3B68404ADB64924D	15
265	8.5473	0000489366D6AC48B59B2D207972A7E5432FE5603796A3C319F08EC07C67388EAAA	19
267	8.6077	1CE3FF1EE6303981FF8EFDAD4D7A68754CC025E7A0C0F8AEDAA959B5366E92AB6C9	15
269	8.5533	00044082B5CD9B863C953DA87B413F38D48FC9353AF1681D3F8D26919CDF028AEEAA	15
271	8.5496	03E68F8220F8E3C441F95788998DA34F71922C378D99DDA01A9444B6DAD775ADE6B5	13
273	8.8221	00F0B85C0C72596C846774F322D11A054022AFEA1BBC2334F766E8C79E36CADE90B4A	15
275	8.5607	010183C7020637F8E56E0CC10AF4279B64CC639A742FD14CD6E06DAA3657524B59515	17
277	8.6058	01F06C2C91741C9B351770FEC09E48014D889CFAA8E58AC54B77BF318DAF7B8C2C6B5A	15
279	8.5898	01DFE083C100C62740C4BA7F371E6249A1EE979C7669232A7BC4D42764D514B5D6A895	17
281	8.5763	0756A543AB7350A4E93891673AA8A3ACB69CD870C120801367B89384E0BF37012FE07F6	17
283	8.2925	000031BF9CC3A578A735BB90ED29E49F8E4EC6DA9C69F0ED1BB8327DA07B4C9AB935555	17
285	8.5789	00C91658B0F1264C581A103A2FE15F6CAFCC6CD40C75FA54212BA1A9ECE63B4B09E7B8CA	15
287	8.3184	1E6B3C946F28C7790A9C847B7B2B02A302AFAFF537F53F3A3A45C9FD1A24DF2E41CB3E69	17
289	8.1436	01907C03C37F71DF96C241D101BE8C97048EE448EB78C851ABBDAE2C795DB7572D2AD6B9A	15
291	8.1912	00FBF8380E3F809C03137D67359013846696CE1E645B15183260A313549D5AB6D5B5ABAD5	19
293	8.0898	007843C21087843C09FFC1373B28EB311D6668667DBB304831373AD558AD2E968BA2D2E96A	17
295	8.0534	0780067FF006078033803C307E73B739CD9685E18C9B23B26A534B55B355A56552AA6555A5	17
297	8.0190	07145E287D1AAAA8FA14F48F29B836694ECCC32CCC4F8672918348F4FA1480001BD6825EFB6	17
299	8.1496	01FFEF631BD2188039CCC87C22C3FACF4E9A6BE79EC2CFAB4F74A5CCC9B55D970B9362EAA95	15
301	8.3304	01F27C3817FF00910EDFB6F8C386C64E77537DDD73F764E6C692C94715C4BB8AB557A92D635A	17
303	8.0718	14A29CEB1AEA5CA5A43D99B1250A3155FEF3433432EA80137D0713998B4787C87EF93EC9F7C1	21
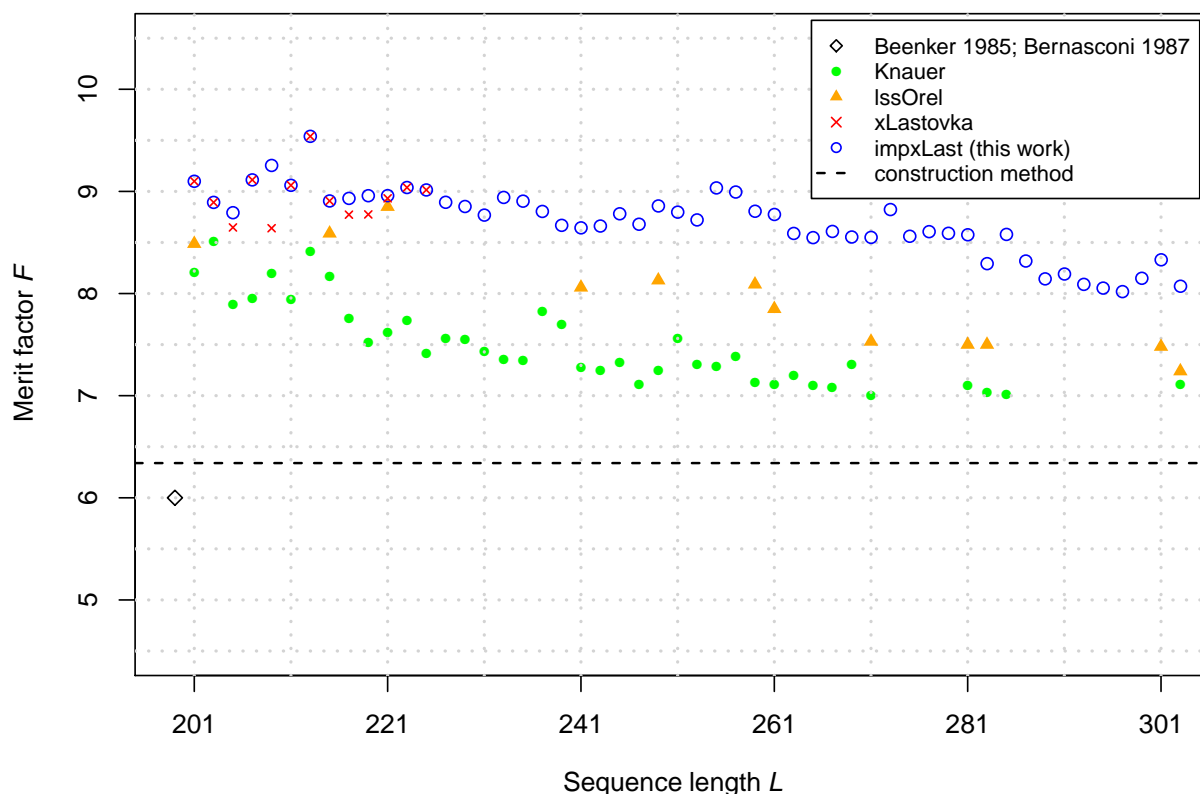
Figure 2: Merit factor values for skew-symmetric binary sequences with odd length from 201 up to 303. Beenker & Bernasconi's result is for length 199.

Table 1 presents skew-symmetric sequences of lengths 201 up to 303. For each length L, a sequence is presented using a hexadecimal notation. We decode each hexadecimal digit in binary form ($0 \mapsto 0000$, $1 \mapsto 0001$, $2 \mapsto 0010$, ..., F $\mapsto 1111$), and, if necessary, remove initial 0 symbols to obtain a binary string of the appropriate length. Then we convert each 0 to $+1$ and each 1 to -1 to obtain the binary sequence. Table 1 contains binary sequences with merit factor $F > 8$ and some of them have $F > 9$. In this table, we show also the results up to $L = 225$ that are taken from our previous work [7], except for seven sequences with odd lengths of 205, 209, 215–221, where the merit factors have been improved by the impxLast algorithm. For all other sequences longer or equal than 227 we report the merit factor values that are obtained by the impxLast and these results are currently the best known.

The largest sequence with a merit factor higher than 9 found to date has a length of 255. Interestingly, it could be pointed out that all sequences with $L <= 285$, except for $L = 283$, have the merit factor higher than 8.5. One can notice that currently known best merit factors decrease as lengths of sequences become larger, and this decreasing trend can be seen by results

obtained by our algorithm, but also by results of the lssOrel algorithm, as well as by Knauer's results. The reason for this lies in the fact that the search space is increasing exponentially.

The authors believe that binary sequences with even higher merit factors exist, but one needs even more computational power to find them. Another possibility for searching new sequences with the higher merit factors is to invent new algorithms.

In Table 1, a PSL value for each skew-symmetric binary sequence is shown in the last column. These PSL values are worse compared to the best-known PSL values (see [16, 7]) for sequences with lengths $201 \le L \le 303$. This means that a binary sequence with a high merit factor has a PSL value that is higher (i.e. worse) than the best-known PSL value, and vice versa. Consequently, a designer of a new algorithm should take care of this fact.

5 Conclusion

In this paper, we used a stochastic algorithm and a high-performance computation to search for aperiodic binary sequences with low autocorrelation properties. We reported skew-symmetric binary sequences

of length from 201 up to 303 that have the merit factor F greater than 8, and many of them have $F > 8.5$ and also $F > 9$. The longest sequence with $F > 9$ found to date has length $L = 255$. In the future, our research will focus on searching new longer sequences with higher merit factors using parallel computation on graphical processing units (GPUs). An implementation of a search algorithm using quantum operations [47] is also a possible direction of further research.

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